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On the adiabatic perturbation theory for systems with impacts☆

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Abstract

A basis for the applicability of the formal scheme of adiabatic perturbation theory for systems with impacts is given using the example of three well-known problems, namely, a small sphere between slowly moving walls, rays in a smoothly irregular waveguide with reflecting walls, and an adiabatic piston.

The adiabatic perturbation theory (APT) (see, for example, Ref. 1, pp. 200–211) is used to give an approximate description of the dynamics of smooth Hamiltonian systems, containing fast and slow variables. Estimates have been made of the accuracy of the approximations which this theory provides. The APT procedure can also be formally employed in a number of cases for systems with a discontinuous Hamiltonian, in particular, for systems with impacts. However, the validity of this formal approach does not follow from the existing results on the accuracy of APT for smooth systems. A distinctive feature of the problems considered is the fact that the slow variables vary rapidly (instantaneously) at an impact.

Below we obtain estimates of the accuracy of APT for systems with impacts. Three model problems are considered, which pertain to three main types of systems, for which APT is used, in order to demonstrate the effectiveness of an approach based on APT; the dynamics of these problems is well known and has been described previously using other methods (see, in particular[2] for the problem of an adiabatic piston).

The first approximation of the APT procedure leads to the conclusion that an adiabatic invariant (an approximate integral) is present in the system. This conclusion has also been repeatedly used for systems with reflections, but its validity needs to be verified by direct calculations (compare with, Ref. 3, Section 52). Higher approximations of the APT procedure for systems with impacts were considered formally in Ref. 4. Perturbation theory for non-smooth Hamiltonian systems has only been considered in the case when the phase variables are continuous.[5,6] In systems with impacts some of the phase variables suffer a discontinuity at an impact.

In systems with impacts, in order to obtain results similar to those obtained for smooth systems using higher approximations of APT, a Poincaré map is usually considered. It may turn out to be smooth; then either the perturbation-theory procedure for smooth maps is applied to it (see, for example[7]), or an artificial approach is used, namely, the map is described as a Poincaré map for a certain auxiliary smooth Hamiltonian system, and the procedure of perturbation theory is applied to this auxiliary system.[8]

The advantage of the approach proposed below is that it enables one to operate directly with an initial Hamiltonian system, rather than with the corresponding Poincaré map (which may turn out to be non-smooth as, for example, in the

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problem of an adiabatic piston, Section 3), and it also enables one to consider systems with impacts uniformly with smooth systems and makes calculations more concise.

1. The Fermi-Ulam model

The problem of the oscillations of a particle between two parallel elastically reflecting walls has been called the Fermi-Ulam problem (or model) (Fig. 1).[9] We will assume (for simplicity) that the left wall is at rest while the right wall changes its position slowly. The distance between the walls $d(\varepsilon t) \geq \text{const} > 0$, where t is the time and $\varepsilon > 0$ is a small parameter. We will introduce the slow time variable τ where $\tau = \varepsilon t$. We will assume the function $d(\cdot)$ to be infinitely differentiable. We will consider the motion in a section of slow time which is either independent of ε or increases as ε decreases. In the latter case we will assume that the function d and its derivatives up to any preassigned order are uniformly bounded on the real axis. We will put the mass of the particle equal to unity, in which case the velocity of the particle is its momentum. The dynamics of the particle is determined as follows: between collisions with the walls the particle has a constant velocity v, which changes sign on collision with the wall at rest (the left wall), and when it collides with the moving (right) wall is converted in accordance with the relation $v_1 = 2\dot{d} - v$. The Hamiltonian of the problem is defined everywhere, apart from the walls, and is the Hamiltonian of a freely moving particle with momentum v.

This problem can be interpreted intuitively as a problem of motion in a potential field, the potential of which is equal to zero between the walls and infinity in the remaining part. Since the wall moves slowly, it is reasonable to consider the problem for a frozen position of the wall ($d = \text{const}$). The phase portrait of this problem is shown in Fig. 2. We can

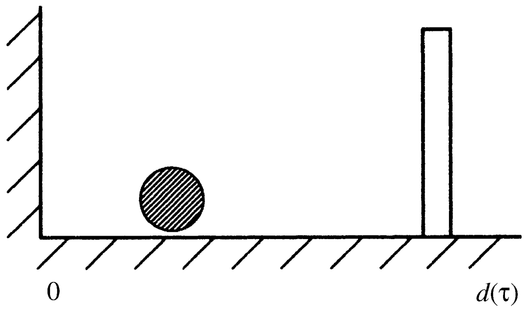

Fig. 1.

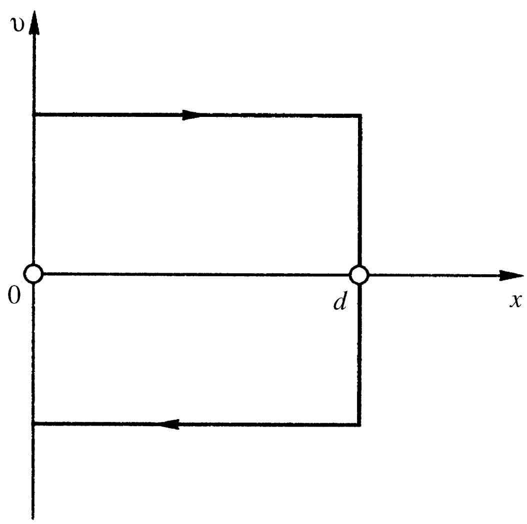

Fig. 2.

introduce "action-angle" variables (I, ϕ) in a standard way for this portrait. The action I is the area bounded by the phase trajectory, divided by 2π:

$$I = \frac{1}{2\pi}|v|2d = \frac{d}{\pi}\sqrt{2E}$$

where E is the Hamiltonian of the particle so that $E=\pi^2I^2/(2d^2)$. The angle (phase) ϕ is the angular variable on the trajectory, which varies uniformly with time, $\phi=2\pi t/T$, where t is the time of motion from the initial position of the particle to the specified position, and T is the period.

If the origin of the phase is chosen to be on the left wall, we have

$$\phi = \begin{cases} \pi x/d, & v>0 \\ \pi(2-x/d), & v<0 \end{cases} \tag{1.1}$$

where x is the distance of the particle from the left wall.

For a standard definition of the phase (1.1), the generating function $W=W(x, I, d)$ for changing from the variables (v, x) to the "action-angle" variables has the form

$$W = \begin{cases} \pi Ix/d, & \phi\in(0,\pi) \\ \pi I(2-x/d), & \phi\in(\pi,2\pi) \end{cases} \tag{1.2}$$

We will make a change of variables defined by the generating function W in the problem with a moving wall. The motion between the walls will be described by a Hamiltonian system with Hamiltonian

$$H = E+\frac{\partial W}{\partial t} = \frac{\pi^2I^2}{2d^2}-\frac{I\dot{d}}{d}f(\phi);\quad f(\phi) = \begin{cases} \phi, & \phi\in(0,\pi) \\ \phi-2\pi, & \phi\in(\pi,2\pi) \end{cases} \tag{1.3}$$

Hamiltonian (1.3) completely describes the motion of a particle between collisions with the walls.

When there is a collision with a fixed wall, the value of the "action" variable does not change. When there is a collision with the moving wall, as direct calculation shows, the change in the "action" variable is described by the relation

$$I_+-I_- = -2d\dot{d}/\pi \tag{1.4}$$

where I_- and I_+ are the values of I at the wall before and after the collision.

We will show how one can describe a collision using Hamiltonian (1.3). Interaction with the wall occurs instantaneously. Hence, the following method will lead to the correct value of the "action" variable after a collision with the moving wall: in Hamiltonian (1.3) we record the instant of time at which a collision occurs, we thereby obtain an integrable Hamiltonian system with one degree of freedom and, using the law of conservation of energy, we calculate the required value.

Lemma 1. *The proposed method defines the correct rule for the change in the "action" variable when collision occurs with a moving wall.*

Proof. We fix the instant of collision and equate the values of the energy of the particle before and after the collision:

$$\frac{\pi^2I_-^2}{2d^2}-\frac{\pi I_-\dot{d}}{d} = \frac{\pi^2I_+^2}{2d^2}+\frac{\pi I_+\dot{d}}{d}$$

We obtain

$$\frac{\pi(I_+^2-I_-^2)}{2d} = -\dot{d}(I_++I_-)$$

whence Eq. (1.4) immediately follows.

The main assertion of the adiabatic perturbation theory, as it applies to the problem in question, is as follows. □

Theorem 1. *By the canonical change of variables*

$$(I, \phi) \mapsto (\hat{I}, \hat{\phi})$$

with a generating function of the form

$$W = \hat{I}\phi + \varepsilon S(\hat{I}, \phi, \tau, \varepsilon); \quad S = S_1 + \varepsilon S_2 + \ldots + \varepsilon^{r-2} S_{r-1}, \quad S_i = S_i(\hat{I}, \phi, \tau) \tag{1.5}$$

where r is any prefixed natural number, Hamiltonian (1.3) takes the form

$$\mathcal{H} = \mathcal{H}_{\Sigma, r}(\hat{I}, \tau, \varepsilon) + \varepsilon^r H_r(\hat{I}, \phi(\hat{I}, \hat{\phi}, \tau, \varepsilon), \tau, \varepsilon); \quad \mathcal{H}_{\Sigma, r} = E + \varepsilon\mathcal{H}_1 + \ldots + \varepsilon^{r-1}\mathcal{H}_{r-1} \tag{1.6}$$

The functions S_i, $\mathcal{H}_i$ are infinitely differentiable with respect to $\hat{I}$, τ and the functions S_i are continuous in ϕ.

Proof. We will use the standard adiabatic perturbation theory procedure (see, for example[1]). In the system with Hamiltonian (1.3) we will make the canonical change of variables with generating function W of the form (1.5). The formulae for transforming the variables are given by the expressions

$$I = \hat{I} + \varepsilon\frac{\partial S}{\partial \phi}, \quad \hat{\phi} = \phi + \varepsilon\frac{\partial S}{\partial \hat{I}} \tag{1.7}$$

The new Hamiltonian is given by the formula

$$\mathcal{H} = H(I, \phi, \tau, \varepsilon) + \varepsilon^2\frac{\partial S(\hat{I}, \phi, \tau, \varepsilon)}{\partial \tau} \tag{1.8}$$

We will choose the functions S_i such that the new Hamiltonian has the form (1.6). Substituting expressions (1.7) into (1.8) and equating terms of like powers in ε, we obtain a system of equations for determining S_i and $\mathcal{H}_i$. For example, for first-order terms in ε we obtain

$$\mathcal{H}_1(\hat{I}, \tau) = \frac{\partial E}{\partial I}\frac{\partial S_1}{\partial \phi} - \frac{Id'}{d}f(\phi)$$

The prime denotes a derivative with respect to τ. Averaging both sides over ϕ, we obtain that $\mathcal{H}_1 = 0$. The function S_1 is now defined by a quadrature and is continuous in ϕ. We can choose the function S_1 so that its average will be equal to zero. We can similarly determine the remaining S_i as functions continuous in ϕ and infinitely differentiable with respect to $\hat{I}$ and τ. Hence the required properties of smoothness for $\mathcal{H}_i$ and H_r also follow. □

Corollary 1.1. *The value of the variable $\hat{I}$ is preserved along the motion with accuracy $O(\varepsilon^r)$ in the time interval $O(\varepsilon^{-k})$ for any prefixed natural value of k. The value of the variable $\hat{\phi}$ is determined by the integrable system with Hamiltonian $\mathcal{H}_{\Sigma,r}$ with an accuracy $O(\varepsilon^{r-k})$ in the same time interval.*

Proof. Consider the change of variables $(I, \phi) \mapsto (\tilde{I}, \tilde{\phi})$ with the generating function

$$\tilde{W} = \tilde{I}\phi + \varepsilon\tilde{S}(\tilde{I}, \phi, \tau, \varepsilon); \quad \tilde{S} = S_1 + \varepsilon S_2 + \ldots + \varepsilon^{r+k-2}S_{r+k-1}$$

It follows from Theorem 1 that the Hamiltonian in the new variables takes the form

$$\tilde{\mathcal{H}} = \mathcal{H}_{\Sigma, r+k}(\tilde{I}, \tau, \varepsilon) + \varepsilon^{r+k}H_{r+k}(\tilde{I}, \phi, \tau, \varepsilon) \tag{1.9}$$

It follows from the formulae for the change of variables that $\hat{I} - \tilde{I} = O(\varepsilon^r)$. Hence, below we will consider the change in the values of the variables $\tilde{I}$ and $\hat{\phi}$. The change in the value of $\tilde{I}$ between successive collisions of the particle with the walls is a quantity $O(\varepsilon^{r+k})$. We will consider the variation of the variable $\tilde{I}$ at the instant when the particle collides with the moving wall. In Lemma 1 a rule is given for calculating the value of I after the particle collides with the wall. In order to calculate the change in the value $\tilde{I}$ using this rule, we substitute the expression

$I = I(\tilde{I}, \phi, \tau, \varepsilon)$ into Hamiltonian H (1.3) and take into account the fact that $H = \mathcal{H} - \varepsilon^2 \partial \tilde{S}/\partial\tau$. Hence, we obtain the relation

$$\mathcal{H}_{\Sigma, r+k}(\tilde{I}_+, \tau, \varepsilon) + \varepsilon^{r+k} H_{r+k}(\tilde{I}_+, \pi + 0, \tau, \varepsilon) - \varepsilon^2 \partial \tilde{S}(\tilde{I}_+, \pi, \tau, \varepsilon)/\partial\tau =$$
$$= \mathcal{H}_{\Sigma, r+k}(\tilde{I}_-, \tau, \varepsilon) + \varepsilon^{r+k} H_{r+k}(\tilde{I}_-, \pi - 0, \tau, \varepsilon) - \varepsilon^2 \partial \tilde{S}(\tilde{I}_-, \pi, \tau, \varepsilon)/\partial\tau$$

where $\tilde{I}_-$ and $\tilde{I}_+$ are the values of $\tilde{I}$ at the wall before and after impact; we have taken into account the fact that the function $\tilde{S}$ is continuous in ϕ. Hence, we obtain

$$(\partial E(\tilde{I}_-, \tau)/\partial \tilde{I} + O(\varepsilon^2))(\tilde{I}_+ - \tilde{I}_-) = O(\varepsilon^{r+k})$$

such that $\tilde{I}_+ - \tilde{I}_- = O(\varepsilon^{r+k})$

The number of collisions a particle makes with the walls in a given time interval is estimated by a quantity $O(\varepsilon^{-k})$. Hence, due to collisions, the value of $\tilde{I}$ may vary only by $O(\varepsilon^r)$. The part of the assertion which touches on the value of the variable $\tilde{I}$ is proved.

Consider the change in the variable $\hat{\phi}$. In each time interval between successive collisions of a particle with the moving wall, the deviation of the value of the variable $\hat{\phi}$ from the solution of the equation

$$\dot{\hat{\phi}} = \partial \mathcal{H}_{\Sigma, r}/\partial \hat{I}, \quad \hat{I} = \text{const}$$

amounts to $O(\varepsilon^r)$. In a collision with the moving wall the value of the variable $\hat{\phi}$ undergoes a jump, which is determined by the change in the value of the variable $\hat{I}$ from the formulae for the change of variables (1.7). Since S_m are continuous functions of ϕ, the change in the value of $\hat{\phi}$ when there is a collision with the moving wall is a quantity $O(\varepsilon^{r+1})$, which completes the proof. □

Corollary 1.2. *The value of the variable I is preserved with an accuracy $O(\varepsilon)$ in a time interval $O(\varepsilon^{-k})$ for any prefixed natural k.*

Corollary 1.3. *The behaviour of the variables I and ϕ in an interval $O(\varepsilon^{-k})$ is determined by the formulae for the change of variables (1.5) together with the approximate formulae*

$$\hat{I} = \text{const}, \quad \dot{\hat{\phi}} = \frac{\partial \mathcal{H}_{\Sigma, r}}{\partial \hat{I}}$$

with an accuracy of $O(\varepsilon^r)$ for the variable I and $O(\varepsilon^{r-k})$ for the variable ϕ.

Remark 1. Lemma 1, Theorem 1 and its Corollaries 1.1–1.3 can be extended, without any change in formulation, to the case when a force field with an infinitely differentiable potential $U(x, \tau)$ exists between the walls, and the motion of the particle in the adiabatic approximation is such that collision with the walls occurs with a velocity which differs from zero by a positive constant.

Remark 2. By a canonical change of variables

$$I, \phi \mapsto \hat{I}, \phi; \quad I = \hat{I} + \frac{dd'}{\pi^2} f(\phi)$$

the Fermi–Ulam problem can be reduced to an investigation of a Hamiltonian system with Hamiltonian

$$\hat{H} = \frac{\pi^2 \hat{I}^2}{2d^2} + \frac{d\ddot{d}}{2\pi^2} f^2(\phi) \tag{1.10}$$

The function $f^2(\phi)$ is continuous in ϕ. When there is a force field between the walls it is hardly possible to reduce the system to a system with a continuous Hamiltonian.

Remark 3. The result of Lemma 1, including the case when there is a force field between the walls, can be derived from the elastic collision law, as pointed out to us by Ye.I. Kugushev, and from the variational principle, as mentioned by S.V. Bolotin.

Remark 4. When investigating the dynamics of systems with impacts, the following approach has been used systematically.[10] The system with impacts is replaced by a smooth system with a high repulsive potential, which acts in a thin layer (the layer thickness $\delta \ll 1$, and the potential gradient $\sim 1/\delta$). The properties of the system with impacts are derived from the properties of a smooth system by taking the limit as $\delta \to 0$. Following this approach, we could define the potential in the Fermi–Ulam problem (and in the problems considered below in Sections 2 and 3), to carry out the procedure of perturbation theory for a smooth system and to derive estimates of the accuracy of this procedure for a system with impacts from corresponding estimates for the smooth system. But it would then be necessary to consider the problem of the uniformity of the estimates over δ. The approach employed above enables us to avoid this problem.

2. A light ray in a plane waveguide

In this section we will consider the problem of the trajectory of a light ray in a plane waveguide with reflecting walls. The waveguide is assumed to be smoothly irregular[11] (i.e. its width varies slowly in the ray direction). For simplicity we will assume that the lower wall of the waveguide coincides with the x axis. The position of the upper wall in the Oxy plane is given by the formula $y = d(X)$, where $X = \varepsilon x$, and ε is a small parameter (Fig. 3). We assume the function $d(\cdot)$ to be infinitely differentiable. The motion is considered in a time interval $O(\varepsilon^{-k})$, where k is a prefixed natural number. We assume that the function d, its derivatives up to any preassigned order and the function $1/d$ are uniformly bounded on the whole real axis.

The ray propagation in a medium is described by a Hamiltonian system with Hamiltonian

$$H = p_x^2 + p_y^2 - n^2(x, y) \tag{2.1}$$

where we must consider only the energy level $H = 0$.[11] Here $p_{x,y}$ are variables, canonically conjugate to the coordinates x, y, and $n(x, y)$ is the refractive index of the medium. In the case considered, $n = 1$ inside the waveguide, while on the reflection walls the Hamiltonian system is not defined and only the conservation laws can be used.

In the unperturbed system $X = \text{const}$, the projection of the phase trajectory onto the p_y, y plane has the same form as the phase trajectory of the particle in Fig. 2. As in the previous section, instead of the variables (p_y, y) we can introduce "action-angle" variables (I, ϕ): $I = |p_y| d(X)/\pi$; if the origin of the phase is chosen to be on the lower wall, we have

$$\phi = \begin{cases} \pi y/d(X), & p_y > 0 \\ \pi(2 - y/d(X)), & p_y < 0 \end{cases} \tag{2.2}$$

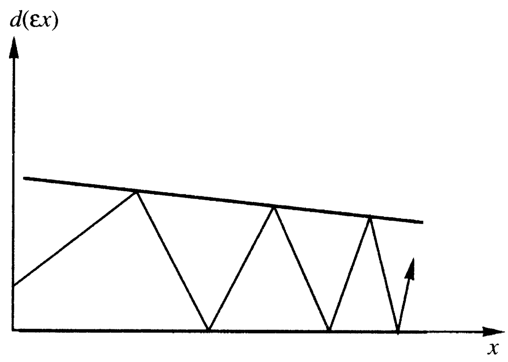

Fig. 3.

With the standard definition of the phase (2.2) the generating function for changing from the variables (p_y, y) to the "action-angle" variables has the form

$$W(I, y, X) = \begin{cases} \pi I y/d, & \phi \in (0, \pi) \\ \pi I(2 - y/d), & \phi \in (\pi, 2\pi) \end{cases} \tag{2.3}$$

In the complete (unperturbed) system, we will make the following canonical change of variables

$$(p_x, x, p_y, y) \mapsto (\hat{p}_x, x, I, \phi)$$

with generating function

$$S = \hat{p}_x x + W(I, y, \varepsilon x)$$

The Hamiltonian in the new variables takes the form

$$H = \frac{\pi^2 I^2}{d^2} + \left(\hat{p}_x - \varepsilon \frac{Id'}{d} f(\phi)\right)^2 - 1 \tag{2.4}$$

The function $f(\phi)$ is given by the last formula of (1.3) and the prime denotes differentiation with respect to X.

The value of the "action" variable does not change on reflection from the lower wall. For reflection from the upper wall, as direct calculation shows, the change in the "action" variable is described by the relation

$$I_1 = I - \frac{2\varepsilon d'}{1 + \varepsilon^2 d'^2}\left(\varepsilon d' I + \frac{dp_x}{\pi}\right) \tag{2.5}$$

Remark 5. The quantity $\hat{p}_x$ does not change on reflection. Formula (2.5) describes the conversion of the angle at which the ray is incident on the lower wall of the waveguide. This angle, after reflection of the ray at the upper wall, is converted as given by

$$\alpha_1 = \alpha - 2\,\mathrm{arctg}(\varepsilon d')$$

whence relation (2.5) follows.

Lemma 2. *If we fix the variables $(\hat{p}_x, x)$ in Hamiltonian (2.4) and consider the system obtained with one degree of freedom, the condition for the value of the Hamiltonian to remain the same on passing through the point $\phi = \pi$ gives the correct change in the "action" variable I when the ray is reflected from the upper wall of the waveguide.*

Proof. We equate the values of Hamiltonian (2.4) before and after the reflection of the ray from the upper wall. After reduction we obtain

$$\frac{\pi^2 I^2}{d^2}(1 + \varepsilon^2 d'^2) - 2\varepsilon \hat{p}_x \frac{Id'}{d}\pi = \frac{\pi^2 I_1^2}{d^2}(1 + \varepsilon^2 d'^2) + 2\varepsilon \hat{p}_x \frac{I_1 d'}{d}\pi$$

Since the value of the "action" variable is a positive quantity, we have

$$I_1 = I - \frac{2\varepsilon d'}{1 + \varepsilon^2 d'^2}\frac{\hat{p}_x d}{\pi}$$

Substituting here the expression for the momentum

$$p_x = \hat{p}_x - \varepsilon \pi I d'/d$$

we obtain expression (2.5). □

We will formulate the main assertion of adiabatic perturbation theory as it applies to the problem being considered.

Theorem 2. *By the canonical change of variables*

$$(I, \phi, \hat{p}_x, x) \mapsto (\tilde{I}, \tilde{\phi}, \tilde{p}_x, \tilde{x}) \tag{2.6}$$

with generating function of the form

$$W = \tilde{I}\phi + \tilde{p}_x x + \varepsilon S(\tilde{I}, \phi, \tilde{p}_x, \varepsilon x; \varepsilon); \quad S = S_1 + \varepsilon S_2 + \ldots + \varepsilon^{r-2} S_{r-1} \tag{2.7}$$

where r is any prefixed natural number, Hamiltonian (2.4) can be reduced to the form

$$\begin{aligned} H &= \mathcal{H}_{\Sigma, r}(\tilde{I}, \tilde{p}_x, \varepsilon\tilde{x}, \varepsilon) + \varepsilon^r H_r(\tilde{I}, \phi, \tilde{p}_x, \varepsilon x, \varepsilon) \\ \mathcal{H}_{\Sigma, r} &= \pi^2 \tilde{I}^2/d^2 + \tilde{p}_x^2 - 1 + \varepsilon\mathcal{H}_1 + \ldots + \varepsilon^{r-1}\mathcal{H}_{r-1} \end{aligned} \tag{2.8}$$

The functions S_i and $\mathcal{H}_i$ are infinitely differentiable with respect to $\tilde{I}$, $\tilde{p}_x$, $\varepsilon\tilde{x}$, and the functions S_i are continuous in ϕ.

Proof. We make the canonical change of variables (2.6) with generating function (2.7). We write the formulae for the change of the variables

$$I = \tilde{I} + \varepsilon\frac{\partial S}{\partial \phi}, \quad \tilde{\phi} = \phi + \varepsilon\frac{\partial S}{\partial I}, \quad \hat{p}_x = \tilde{p}_x + \varepsilon^2\frac{\partial S}{\partial \varepsilon x}, \quad \varepsilon\tilde{x} = \varepsilon x + \varepsilon^2\frac{\partial S}{\partial \tilde{p}_x} \tag{2.9}$$

Acting in the standard way in adiabatic perturbation theory, we can determine the functions S_m that are continuous and periodic in ϕ so as to eliminate the dependence of the Hamiltonian on ϕ from the terms $O(\varepsilon^m)$. The procedure is as follows: we substitute the formulae for change of the action and longitudinal variables $(\hat{p}_x, x)$ from formulae (2.9) into Hamiltonian (2.4) and require that, in the new variables, the Hamiltonian acquires the form (2.8). The functions S_m will be determined recurrently. For example, for S_1 we have the equation

$$\frac{\partial S_1}{\partial \phi} = \frac{\tilde{p}_x d(X)d'}{\pi^2} f(\phi)$$

The functions S_m can, for example, be chosen so that they have a zero mean over ϕ. Finally, the Hamiltonian is converted to the necessary form. The functions S_i and $\mathcal{H}_i$ obviously possess all the properties mentioned in the conditions of the theorem.

Consider the solutions of Hamiltonian systems with Hamiltonians (2.8) and $\mathcal{H}_{\Sigma,r}(J, p, \varepsilon x, \varepsilon)$ with the initial conditions

$$\tilde{I}(0) = J(0), \quad \tilde{\phi}(0) = \psi(0), \quad \tilde{p}_x(0) = p(0), \quad \tilde{x}(0) = x(0)$$

where ψ is a variable conjugate to the variable J. □

Corollary 2.1. *The value of the variable $\tilde{I}$ is preserved with an accuracy $O(\varepsilon^r)$ in the time interval ε^{-k} for any prefixed natural value of k. The projection of the ray trajectory onto the $\tilde{p}_x$, $\varepsilon\tilde{x}$ plane for this time interval lies in the $O(\varepsilon^r)$ neighbourhood of the curve $\mathcal{H}_{\Sigma,r}(J, p, \varepsilon x, \varepsilon) = 0$. The behaviour of the variables $\varepsilon\tilde{x}$, $\tilde{p}_x$, $\tilde{\phi}$ is described by the solution of a Hamiltonian system with Hamiltonian $\mathcal{H}_{\Sigma,r}$ with accuracies of $O(\varepsilon^{r-k+1})$, $O(\varepsilon^{r-k+1})$ and $O(\varepsilon^{r-k})$ respectively (under certain natural additional conditions indicated below).*

Proof. Using a similar approach to that described in the previous section, we will consider an auxiliary change of the initial variables with generating function

$$W = \bar{I}\phi + \bar{p}_x x + \varepsilon S_1(\bar{I}, \phi, \bar{p}_x, \varepsilon x) + \ldots + \varepsilon^{r+k-1} S_{r+k-1}(\bar{I}, \phi, \bar{p}_x, \varepsilon x)$$

Hamiltonian (2.4) will then, according to Theorem 2, reduce to the form

$$H = \mathcal{H}_{\Sigma, r+k}(\bar{I}, \bar{p}_x, \varepsilon\bar{x}, \varepsilon) + \varepsilon^{r+k} H_{r+k}(\bar{I}, \phi, \bar{p}_x, \varepsilon x, \varepsilon) \tag{2.10}$$

where ϕ and x are considered as functions of the new variables $(\bar{I}, \bar{\phi}, \bar{p}_x, \bar{x})$. It follows from the formulae for the change of variables that

$$\tilde{I} - \bar{I} = O(\varepsilon^r), \quad \tilde{\phi} - \bar{\phi} = O(\varepsilon^r), \quad \tilde{x} - \bar{x} = O(\varepsilon^r), \quad \tilde{p}_x - \bar{p}_x = O(\varepsilon^{r+1})$$

It is therefore sufficient to consider the behaviour of the variables $\bar{I}$, $\bar{\phi}$, $\bar{p}_x$, $\bar{x}$. □

We will consider the question of the accuracy with which the value of the variable $\tilde{I}$ is maintained. In each time interval between successive reflections of the ray from the upper wall of the waveguide, the change in the variable $\tilde{I}$ is given by Hamilton's equation

$$\dot{\bar{I}} = -\varepsilon^{r+k}\partial H_{r+k}/\partial\bar{\phi}$$

which, on summing over all such intervals, gives $O(\varepsilon^r)$.

It follows from expression (2.10) that the values of the variables $(\bar{p}_x, \bar{x})$ undergo a discontinuity when the ray is reflected from the upper wall. However, the generating function for the change of variables is a continuous function of ϕ. Hence, it follows from the formulae for the change of variables that

$$\Delta\bar{p}_x = O(\varepsilon^2\Delta\bar{I}), \quad \Delta\varepsilon\bar{x} = O(\varepsilon^2\Delta\bar{I})$$

where $\Delta\bar{I}$, $\Delta\bar{p}_x$, $\Delta\varepsilon\bar{x}$ are the jumps of the variables $\bar{I}$, $\bar{p}_x$, $\varepsilon\bar{x}$ when the ray is reflected from the upper wall of the waveguide.

We can now conclude on the basis of Lemma 2 that the change in the value of the variable $\bar{I}$ when the ray is reflected from the upper wall of the waveguide is $O(\varepsilon^{r+k})$. The number of reflections of the ray from the upper wall in a time interval of the order of ε^{-k} is $O(\varepsilon^{-k})$, and hence the complete change in the value of the variable $\bar{I}$ is a quantity $O(\varepsilon^r)$, and the part of the assertion touching on the variable $\bar{I}$ is proved.

The projection of the trajectory of the ray on the $\tilde{p}_x$, $\varepsilon\tilde{x}$ plane lies in a $O(\varepsilon^r)$-neighbourhood of the curve

$$\mathcal{H}_{\Sigma,r}(J, p, \varepsilon x, \varepsilon) = 0$$

since along the ray trajectory $\mathcal{H}_{\Sigma,r}(J, \tilde{p}_x, \varepsilon\tilde{x}, \varepsilon) = O(\varepsilon^r)$.

The values of the variables p, x, ψ vary continuously, while the values of the variables $\bar{p}_x$, $\bar{x}$, $\bar{\phi}$ undergo a jump when the ray is reflected from the upper wall of the waveguide. As mentioned above, the jump in the values of the variables $\bar{p}_x$, $\bar{x}$, $\bar{\phi}$ is calculated in terms of the jump in the value of $\bar{I}$ when the ray is reflected from the upper wall as

$$\Delta\bar{p} = O(\varepsilon^2\Delta\bar{I}) = O(\varepsilon^{r+k+2}), \quad \Delta\bar{x} = O(\varepsilon\Delta\bar{I}) = O(\varepsilon^{r+k+1}), \quad \Delta\bar{\phi} = O(\varepsilon\Delta\bar{I}) = O(\varepsilon^{r+k+1})$$

Hence, the total change in the values of the variables $(\bar{p}_x, \bar{x}, \bar{\phi})$ due to reflections amounts to $O(\varepsilon^{r+2})$ for the variable $\bar{p}_x$ and $O(\varepsilon^{r+1})$ for the variables $\bar{x}$ and $\bar{\phi}$. The total changes in the variables $\tilde{p}_x$, $\tilde{x}$, $\tilde{\phi}$ due to reflections are the same.

Consider the zeroth approximation of the Hamiltonian $\mathcal{H}_{\Sigma,r}$:

$$\mathcal{H}^0_{\Sigma,r} \overset{\text{def}}{=} F = \frac{\pi^2 I^2}{d^2(\varepsilon x)} + p^2$$

The Hamiltonian system with Hamiltonian $\mathcal{H}^0_{\Sigma,r}/2$ describes the motion of a particle in a force field with potential $U = \pi^2 I^2/[2d^2(\varepsilon x)]$. We will assume (see Fig. 4) that for the ray being considered the level of the function F lies either: (1) above all the local maxima of $2U$, or (2) a single reflection of the ray from a potential hump is possible, or (3) motion occurs between potential humps (a resonator).

Consider case 1, when the ray propagates in a specified direction. The variables $\tilde{x}$ and x then vary monotonically with time. We can express the variables $\tilde{p}_x$ and $\tilde{p}$ as functions of the variables $\varepsilon\tilde{x}$ and εx respectively. When $x = \tilde{x}$, the

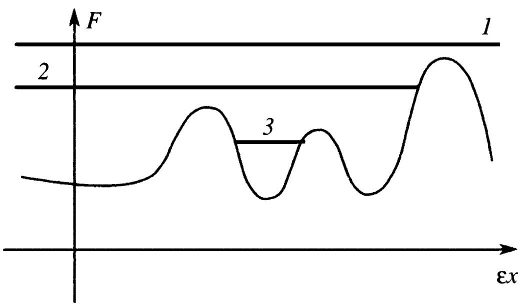

Fig. 4.

values of the variables $\tilde{p}_x$ and p differ by an amount $O(\varepsilon^r)$. The difference in the times at which the variables $\tilde{x}$ and x reach a specified value of x_* is estimated from the formula

$$\int_{x(0)}^{x_*}\left(\frac{1}{\dot{\tilde{x}}}-\frac{1}{\dot{x}}\right)dx = O(\varepsilon^{r-k})$$

Consequently, for a specified instant of time t we have

$$\varepsilon\tilde{x}(t)-\varepsilon x(t) = O(\varepsilon^{r-k+1}), \quad \tilde{p}_x(t)-p(t) = O(\varepsilon^{r-k+1})$$

The difference $\tilde{\phi}-\psi$ when $x=\tilde{x}=x_*$ is

$$\tilde{\phi}-\psi = \int_{x(0)}^{x_*}\left(\frac{\omega(\tilde{I}, \tilde{p}_x, \varepsilon x, \varepsilon)}{\dot{\tilde{x}}}-\frac{\omega(J, p, \varepsilon x, \varepsilon)}{\dot{x}}\right)dx + O(\varepsilon^{r-k}) = O(\varepsilon^{r-k}); \quad \omega = \frac{\partial \mathcal{H}_{\Sigma, r}}{\partial I} \tag{2.11}$$

Consequently, for a specified instant of time t we have

$$\tilde{\phi}(t)-\psi(t) = O(\varepsilon^{r-k})$$

In case 2, when a single change in the direction of the ray propagation is possible, we can use x and p as the monotonically varying variables in different regions. In case 3, when the waveguide has the configuration of a resonator, we can use the "angle" variable on the phase portrait of the system with Hamiltonian $\mathcal{H}_{\Sigma,r}(I, p, x)$ as the monotonically varying variable. In other respects the estimates are the same as in case 1.

Corollary 2.2. *The value of the variable I is preserved with accuracy $O(\varepsilon)$ in the time interval $O(\varepsilon^{-r})$ for any prefixed natural value of k.*

Corollary 2.3. *The behaviour of the variables I, ϕ, $\hat{p}_x$, x is determined from the change in variables (2.6) together with approximate formulae which specify the motion in the system with Hamiltonian $\mathcal{H}_{\Sigma,r}(\tilde{I}, \tilde{p}_x, \varepsilon\tilde{x})$, with the accuracy with which the behaviour of the variables $\tilde{I}$, $\tilde{\phi}$, $\tilde{p}_x$, $\tilde{x}$ is described by these approximate formulae according to Corollary 2.1.*

Remark 6. In a time interval $O(\varepsilon^{-1})$ the ray trajectory is determined, with an accuracy $O(\varepsilon)$, by a Hamiltonian system with a single degree of freedom with Hamiltonian

$$H = \pi^2 I^2/d^2 + \hat{p}_x^2 - 1 \tag{2.12}$$

at the energy level $H=0$.

Remark 7. Lemma 2, Theorem 2 and its Corollaries 2.1–2.3 can be extended without change in the formulation to the case when there is a medium with an infinitely differentiable refraction index $n(\varepsilon x, y)$, differing from unity, between the walls, and the ray propagates in the adiabatic approximation such that, when there is reflection from the wall, the angle between the ray and the wall differs from zero by a positive constant.

Remark 8. The result of Lemma 2, including the case when there is a medium present, can be derived from the variational principle, as was pointed by S. V. Bolotin.

3. The dynamics of a massive piston in a gas of light particles

The problem of an adiabatic piston is an important model problem in statistical mechanics, considered in relation to attempts to derive the laws of thermodynamics from the laws of mechanics (see, for example,[2,12]). In this problem a system consisting of a massive cylindrical piston in a gas of similar light particles (in the sense that the overall mass of the particles is small compared with the mass of the piston) is considered. The particles move independently, undergoing elastic collisions with the walls of the cylindrical vessel in which the system is placed and also with the piston.

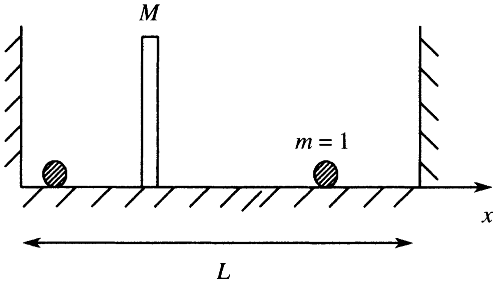

Fig. 5.

The mass of a particle will be assumed to be equal to unity. The vessel length, after subtracting the piston thickness, L, and the number of particles of the gas are assumed to be quantities of the order of unity. The mass of the piston M is assumed to be large compared with the overall mass of the particles. At the initial instant of time the piston is at rest, while the velocities of all the particles are non-zero. The energy of the system is then independent of the mass of the piston. Consequently, the energy of the piston can be estimated as $O(1)$ and its characteristic velocity of motion is estimated as $O(M^{-1/2})$. Hence, it is convenient to introduce a small parameter $\varepsilon = M^{-1/2}$. The time interval in which the investigation is carried out is equal to ε^{-1} in order of magnitude.

Below, for simplicity, we will consider the case when there is only one particle on both sides of the piston (Fig. 5). As will be seen, the discussion can be extended directly to the case of a finite number of particles, and we will finally present the result for the general case. The velocities of the particles will be assumed to be directed along the axis of the vessel, so that the motion of the particles is one-dimensional; this does not limit the generality. We will denote variables belonging to the left and right particles by subscripts l and r respectively. Variables without subscripts refer to the piston. The total energy of the system is given by the formula

$$E = \varepsilon^2 P^2/2 + p_l^2/2 + p_r^2/2 \tag{3.1}$$

where p_l and p_r are the momenta of the particles and P is the momentum of the piston. We will denote by x_l, x_r and X the distances of the particles and the piston from the left wall of the cylinder, while for the right-hand particle the distance will be taken after subtracting the piston thickness.

We will first consider the motion of particles for a fixed piston. For each particle we can change from the variables (p, x) to the "action-angle" variables (I, ϕ) as in the Fermi–Ulam problem (Section 1).

Choosing the origin of the "angle" variables ϕ_l and ϕ_r on the walls of the vessel, we have

$$\phi_l = \psi(x_l, \operatorname{sign} p_l, X) = \begin{cases} \pi x_l/X, & p_l > 0, \\ \pi(2 - x_l/X), & p_l < 0, \end{cases} \quad \phi_r = \psi(L - x_r, -\operatorname{sign} p_r, L - X) \tag{3.2}$$

The "action" variables have the form

$$I_l = |p_l| X/\pi, \quad I_r = |p_r|(L - X)/\pi$$

The generating functions for changing from the (p, x) variables to the "action-angle" variables have the form

$$S_l(I_l, x_l, X) = I_l \psi(x_l, \operatorname{sign} p_l, X), \quad S_r(I_r, x_r, X) = I_r \psi(L - x_r, -\operatorname{sign} p_r, L - X)$$

In the problem of a moving piston we will make the change of variables with generating function

$$W = \hat{P}X + S_l(I_l, x_l, X) + S_r(I_r, x_r, X)$$

and write the formula for the Hamiltonian in the new variables

$$\mathcal{H} = \varepsilon^2 \frac{1}{2}\left(\hat{P} - \frac{I_l}{X} f(\phi_l) + \frac{I_r}{L - X} f(\phi_r)\right)^2 + \frac{\pi^2 I_l^2}{2X^2} + \frac{\pi^2 I_r^2}{2(L - X)^2} \tag{3.3}$$

The function $f(\phi)$ is given by the second formula of (1.3).

In order to obtain the change in the value of the "action" variable of a particle and the momentum of the piston as a result of a collision, it is natural to use the following approach: fix all the variables, apart from the variables describing the motion of the particle participating in a collision, for this particle therefore obtain a system with one degree of freedom, and calculate the jump in the value of the "action" variable from the conservation law for the value of Hamiltonian.

We will formulate the corresponding lemma for the left particle (the formulation is similar for the right particle). The values of the variables after a collision will be denoted by a prime.

Lemma 3. *The proposed method gives the correct value of the change in the "action" variable when there is a collision:*

$$I_l' = I_l - \frac{2PX}{\pi M} + \frac{2}{M+1}\left(\frac{PX}{\pi M} - I_l\right) \tag{3.4}$$

Remark 9. The velocities v and V of particles with masses 1 and M when there is a collision, as is well known, are transformed by the formulae

$$V' = V + \frac{2}{M+1}(v - V), \quad v' = 2V - v + \frac{2}{M+1}(v - V)$$

The "action" of the particle and the momentum of the piston at the instant of collision are written as $I_l = |v|X/\pi$ and $P = MV$, whence formula (3.4) follows.

Remark 10. If two particles simultaneously collide with the piston, the motion after the collision is not defined. The motion in the case of such collisions will not be considered (the corresponding initial data have zero measure).

Proof. A collision is a transition of the phase ϕ_l through the value π. Equating the energy of the system before and after the collision, we obtain

$$\frac{\pi^2 I_l^2}{2X^2}\frac{M+1}{M} - \frac{\pi I_l}{MX}\left(\hat{P} + \frac{I_r f(\phi_r)}{L-X}\right) = \frac{\pi^2 I_l'^2}{2X^2}\frac{M+1}{M} + \frac{\pi I_l'}{MX}\left(\hat{P} + \frac{I_r f(\phi_r)}{L-X}\right) \tag{3.5}$$

After reduction, we obtain the value of the "action" variable

$$I_l' = I_l - \frac{2}{M+1}\frac{X}{\pi}\left(\hat{P} + \frac{I_r f(\phi_r)}{L-X}\right) \tag{3.6}$$

Suppose P is the value of the momentum of the piston before the collision; then

$$I_l' = I_l - \frac{2}{M+1}\frac{X}{\pi}\left(P + \frac{\pi I_l}{X}\right) \tag{3.7}$$

whence expression (3.4) follows. □

Theorem 3. *The values of the variables $I_{l,r}$ are preserved with accuracy $O(\varepsilon)$ in a time interval of the order of ε^{-1}.*

Proof. For the piston we introduce the normalized momentum $\check{P} = \varepsilon \check{P}$ and write the Hamiltonian (3.3) with an accuracy $O(\varepsilon^2)$

$$\mathcal{H} = \frac{\check{P}^2}{2} + \frac{\pi^2 I_l^2}{2X^2} + \frac{\pi^2 I_r^2}{2(L-X)^2} - \varepsilon\check{P}\frac{I_l}{X}f(\phi_l) + \varepsilon\check{P}\frac{I_r}{L-X}f(\phi_r) + O(\varepsilon^2) \tag{3.8}$$

We make the canonical change of variables

$$(I_{l,r}, \phi_{l,r}, \varepsilon^{-1}\check{P}, X) \mapsto (\tilde{I}_{l,r}, \tilde{\phi}_{l,r}, \varepsilon^{-1}\tilde{P}, \tilde{X})$$

which eliminates the dependence of the Hamiltonian on the phases of the particles, up to terms $O(\varepsilon^2)$. The generating function of this change of variables has the form

$$W = \frac{1}{\varepsilon}\tilde{P}X + \tilde{I}_l\phi_l + \tilde{I}_r\phi_r + \varepsilon S_l(\tilde{I}_l, \phi_l, \tilde{P}, X) + \varepsilon S_r(\tilde{I}_r, \phi_r, \tilde{P}, X) \tag{3.9}$$

We substitute the expressions relating the new and old variables,

$$\begin{aligned} I_{l,r} &= \tilde{I}_{l,r} + \varepsilon\frac{\partial S_{l,r}}{\partial\phi_{l,r}}, \quad \tilde{\phi}_{l,r} = \phi_{l,r} + \varepsilon\frac{\partial S_{l,r}}{\partial I_{l,r}} \\ \check{P} &= \tilde{P} + \varepsilon^2\frac{\partial S_l}{\partial X} + \varepsilon^2\frac{\partial S_r}{\partial X}, \quad \tilde{X} = X + \varepsilon^2\frac{\partial S_l}{\partial\tilde{P}} + \varepsilon^2\frac{\partial S_r}{\partial\tilde{P}} \end{aligned} \tag{3.10}$$

into Hamiltonian (3.8) and choose functions $S_{l,r}$ so that the dependence of the Hamiltonian on ϕ_l and ϕ_r in terms $O(\varepsilon)$ is eliminated. We obtain the following equations for the functions $S_{l,r}$

$$\frac{\partial S_l}{\partial\phi_l} = \frac{\tilde{P}Xf(\phi_l)}{\pi^2}, \quad \frac{\partial S_r}{\partial\phi_r} = -\frac{\tilde{P}(L-X)f(\phi_r)}{\pi^2} \tag{3.11}$$

from which it follows that the generating functions $S_{l,r}$ are defined apart from an arbitrary function of $\tilde{I}_{l,r}$, $\tilde{P}$, X. We choose this function to be equal to zero and we call the variables $\tilde{I}_{l,r}$ the "improved action". As can be seen from relations (3.10) and (3.11), $\tilde{\phi}_l = \phi_l$ and $\tilde{\phi}_r = \phi_r$, and the Hamiltonian has a discontinuity when $\tilde{\phi}_{l,r} = \pi$.

If follows from Lemma 3 and formulae (3.10) and (3.11) that when a collision occurs the improved action changes by an amount $O(\varepsilon^2)$. The change in the improved action between successive collisions is also $O(\varepsilon^2)$. The number of collisions in a time interval of length ε^{-1} is $O(\varepsilon^{-1})$. Consequently, in a time interval of the order of ε^{-1} the change in the improved-action variables is $O(\varepsilon)$. The "action" and improved-action variables are related by the formulae for the change of the variables as given by the first equation of (3.10). Hence, the change in the "action" variable in the same time interval is $O(\varepsilon)$, and the theorem is proved. □

Corollary 3.1. *The change in the variables εP and X in a time interval $O(\varepsilon^{-1})$ with an accuracy $O(\varepsilon)$ is described by a Hamiltonian system with Hamiltonian*

$$H = \varepsilon^2\frac{P^2}{2} + \frac{\pi^2}{2X^2}I_l^2 + \frac{\pi^2}{2(L-X)^2}I_r^2, \quad I_{l,r} = \text{const} \tag{3.12}$$

where $I_{l,r}$ are the initial values of the "action" variables.

The problem with any prefixed number of particles is considered in exactly the same way. The "action" variable of each particle is preserved with an accuracy $O(\varepsilon)$ in a time interval $O(\varepsilon^{-1})$. The change in the variables εP and X in this time interval is described with an accuracy $O(\varepsilon)$ by a Hamiltonian system with a Hamiltonian which differs from Hamiltonian (3.12) in that I_l^2 is replaced by $I_{\Sigma,l}^2$ and I_r^2 is replaced by $I_{\Sigma,r}^2$, where $I_{\Sigma,l}^2$ and $I_{\Sigma,r}^2$ are the sums of the squares of the initial values of the "action" variables of the particles on the left and, correspondingly, on the right of the piston. (This result was obtained for the first time by another method in Ref. 2)

Remark 11. In the approximation considered the piston performs oscillations in a potential force field with potential

$$U = \frac{\pi^2}{2X^2}I_{\Sigma,l}^2 + \frac{\pi^2}{2(L-X)^2}I_{\Sigma,r}^2.$$

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